LECTURER NOTES

ON

Engineering Mathematics -1

FOR 1st SEMESTER

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DETERMINANT

INTRODUCTION :

The study of determinants was started by Leibnitz in the concluding portion of seventeenth century. This was latter developed by many mathematician like Cramer, Lagrange, Laplace, Cauchy, Jocobi. Now the determinants are used to study some of aspects of matrices.

Determinant : If the linear equations

$a_1x + b_1 = 0$

and $a_2x + b_2 = 0$

have the same solution, then $\frac{b_1}{a_1} = \frac{b_2}{a_2}$

or $a_1b_2 - a_2b_1 = 0$

The expression $(a_1b_2 - a_2b_1)$ is called a **determinant** and is denoted by symbol.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ or by (a_1b_2) where a_1, a_2, b_1 & b_2 are called the elements of the **determinant**. The elements in the horizontal direction from rows, and those in the vertical direction form **columns**. The determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two coloums. So it is called a **determinant of the second order** and it has 2! = 2 terms in its expansion of which one is positive and other is negative. The diagonal term, or the leading term of the determinant is a_1b_2 whose sign is positive.

Again if the linear equations

$a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ .... (ii)

$a_3x + b_3y + c_3 = 0$ .... (iii)

have the same solutions, we have from the last two equations by cross-multiplication.

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{1}{a_2b_3 - a_3b_2}$$

$$\text{or } x = \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, y = \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$$

These values of x and y must satisfy the first equation. Hence $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$

or $a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1$ is denoted by the symbol

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or by $(a_1b_2c_3)$ and has three rows, and three columns. So it is called a **determinant of the third order** and it has 3! = 6 terms of which three terms are positive, and three terms are negative.

MINORS

Minors : The determinant obtained by suppressing the row and the column in which a particular element occurs is called the minor of that element.

Therefore, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ and that of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and so on.

The minor of any element in a third order determinant is thus a second order determinant.

The minors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ are denoted by $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ respectively.

Hence $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

If Δ stands for the value of the determinant, then

$\Delta = a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3$

Cofactors : The cofactor of any element in a determinant is its coefficient in the expansion of the determinant. It is therefore equal to the corresponding minor with a proper sign.

For calculation of the proper sign to be attached to the minor of the element, one has to consider $(-1)^{i+j}$ and to multiply this sign with the minor of the element a_{ij} where i and j are respecively the row and the column to which the element a_{ij} belongs.

Thus $C_{ij} = (-1)^{i+j} M_{ij}$ Where C_{ij} and M_{ij} are respectively the cofactor and the minor of the element a_{ij}.

The cofactor of any element is generally denoted by the corresponding capital letter.

Thus for the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, cofactor of a_1 is

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_1 is $B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$

that of c_1 is $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(The sign is $(-1)^{1+3} = 1$), and so on.

We see that minors and cofactors are either equal of differ in sign only.

With this notation the determinant may be expanded in the form,

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1A_1 + b_1B_1 + c_1C_1$

Similarly we express $= a_2A_2 + b_2B_2 + c_2C_2$

$=a_3A_3 + b_3B_3 + c_3C_3$

By expanding with respect to the elements of the first column, we can write

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$= a_1A_1 + a_2A_2 + a_3A_3$

Similarly $= b_1B_1 + b_2B_2 + b_3B_3$

$= c_1C_1 + c_2C_2 + c_3C_3$

Thus the determinant can be expressed as the sum of the product of the elements of any row (or column) and the corresponding cofactors of the respective elements of the same row (or column).

PROPERTIES OF DETERMINANT

I. The value of a determinant is unchanged if the rows are written as columns and columns as rows.

If the rows and coloums are interchanged in the determinant of 2nd order $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, the determinant becoems $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Each of the two $= a_1b_2 - a_2b_1$

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

In the third order determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

if the rows and column are interchanged, it

becomes $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta'$ (say)

If Δ is expanded by taking the constituents of the first column and Δ' is expanded by taking the constituents of the first row, then

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $$\Delta' = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$\therefore \Delta = \Delta'$ (since the value of determinant of 2nd orders is unchanged if rows and columns are interchanged).

II. If two adjacent rows and columns of the determinant are interchanged the sign of the determinant is changed but its absolute value remains unaltered.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Δ' has been obtained by interchanging the first and second rows of Δ

Expanding each determinant by the constituents of the first column.

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\text{and } \Delta' = a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix}$$

$$= -a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\left[\text{since} \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} + b_2c_1 - c_2b_1 \text{ and} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + b_1c_2 - b_2c_1\right] = -\Delta$$

In this way it can be proved that only the sign changes if any other two adjacent rows or columns are interchanged.

III. If two rows or columns of a determinant are identical, the determinant vanishes.

$$\text{Let } \Delta_2 = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$$

The first two columns in the determinant are identical. If the first and second columns are interchanged, then the resulting determinant becomes $-\Delta_2$ by II. But since these two columns are identical, the determinant remains unaltered by the interchange.

$\therefore \Delta_2 = -\Delta_2$ or, $2\Delta_2 = 0$

$\therefore \Delta_2 = 0$

IV. If each constitutent in any row or any column is multiplied by the same factor, then the determinant is multiplied by that factor.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The determinant obtained when the constituents of the first row are multiplied by m is

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1A_1 - ma_2A_2 + ma_3A_3$$

$= m\,[a_1A_1 - a_2A_2 + a_3A_3] = m\Delta$

V. If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two other determinants in the determinant.

$$\text{In the determinant} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let $a_1 = t_1 + m_1 + n_1$, $a_2 = t_2 + m_2 + n_2$, $a_3 = t_3 + m_3 + n_3$

Then the given determinant

$$= \begin{vmatrix} t_1+m_1+n_1 & b_1 & c_1 \\ t_2+m_2+n_2 & b_2 & c_2 \\ t_3+m_3+n_3 & b_3 & c_3 \end{vmatrix}$$

$= (t_1 + m_1 + n_1) A_1 - (t_2 + m_2 + n_2) A_2 + (t_3 + m_3 + n_3) A_3$

$= (t_1A_1 - t_2A_2 + t_3A_3) + (m_1A_1 - m_2A_2 + m_3A_3) + (n_1A_1 - n_2A_2 + n_3A_3)$

$$= \begin{vmatrix} t_1 & b_1 & c_1 \\ t_2 & b_2 & c_2 \\ t_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} n_1 & b_1 & c_1 \\ n_2 & b_2 & c_2 \\ n_3 & b_3 & c_3 \end{vmatrix}$$

It can be similarly proved that

$$= \begin{vmatrix} a_1+p_1 & b_1+q_1 & c_1 \\ a_2+p_2 & b_2+q_2 & c_2 \\ a_3+p_3 & b_3+q_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & q_1 & c_1 \\ a_2 & q_2 & c_2 \\ a_3 & q_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ P_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 & c_1 \\ p_2 & q_2 & c_2 \\ p_3 & q_3 & c_3 \end{vmatrix}$$

VI. If the constituents of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns) the value of the determinant remains unaltered.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The determinant obtained, when the constituents of first column are increased by *l* times the second column *m* times the corresponding constituents of the third column is

$$\begin{vmatrix} a_1+lb_1+mc_1 & b_1 & c_1 \\ a_2+lb_2+mc_2 & b_2 & c_2 \\ a_3+lb_3+mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} lb_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \text{(by v)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \text{(by iv)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

Cramer's Rule :

A method is given below for solving three simultaneous linear equations in three unknowns. This method may also be applied to solve 'n' equations in 'n' unknowns.

Consider the system of equations.

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \text{........(1)}$$

Where the coefficients are real.

The coefficient of x, y, z as noted in equations (1) may be used to form the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$, where $\Delta_r; r = 1, 2, 3$ is the determinant obtained from Δ by replacing the r^{th} column by d_1, d_2, d_3.

Example –1 : Find the value of $\begin{vmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \\ \mathbf{8} & \mathbf{1} & \mathbf{3} \end{vmatrix}$

Solution : The value of the given determinant

$$= 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$$

$= 5\ (0 - 2) - 2\ (9 - 16) + 1\ (3 - 0)$

$= - 10 + 14 + 3 = 7$

Example – 2. **Prove that** $\begin{vmatrix} \mathbf{a} & \mathbf{a^2} & \mathbf{a^3} \\ \mathbf{b} & \mathbf{b^2} & \mathbf{b^3} \\ \mathbf{c} & \mathbf{c^2} & \mathbf{c^3} \end{vmatrix}$ **= abc (a – b) (b – c) (c – a)**

Solution : L.H.S. $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (Taking a, b, c, from R_1, R_2, R_3)

$= abc \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$, replacing R_1 by $R_1 - R_2$ and R_2 by $R_2 - R_3$)

$= abc\ (a - b)\ (b - c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ (Taking (a – b) & (b – c) common from R_1 & R_2 respectively)

$$= abc\,(a-b)\,(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix} = abc\,(a-b)\,(b-c)\,(c-a)$$

Assignment

1. Find minors & cofactors of the determinants $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

3. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

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MATRIX

MATRIX AND ITS ORDER

INTRODUCTION :

In modern engineering mathematics matrix theory is used in various areas. It has special relationship with systems of linear equations which occour in many engineering processes.

A matrix is a reactangular array of numbers arranged in rows (horizontal lines) and columns (vertical lines). If there are 'm' rows and 'n' Column's in a matrix, it is called an 'm' by 'n' matrix or a matrix of order m × n. The first letter in mxn denotes the number of rows and the second letter 'n' denotes the number of columns. Generally the capital letters of the alphabet are used to denote matrices and the actual matrix is enclosed in parantheses.

$$\text{Hence } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & a_{23} & -- & a_{2n} \\ a_{31} & a_{32} & a_{33} & -- & a_{3n} \\ -- & -- & -- & -- & -- \\ a_{m1} & a_{m2} & a_{m3} & -- & a_{mn} \end{bmatrix}$$

is a matrix of order m × n and 'a_{ij}' denotes the element in the ith row and jth column. For example a_{23} is the element in the 2[nd] row and third column. Thus the matrix 'A' may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

If m = n, the matrix A is called a square matrix of order n × n (or simply n). Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{bmatrix}$$

is a square matrix of order n. The determinant of order n, $\begin{vmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{vmatrix}$

which is associated with the matrix 'A' is called the determinant of the matrix and is denoted by det A or |A|.

TYPES OF MATRICES WITH EXAMPLES

(a) **Row Matrix :** A matrix of order 1 × n is called a row matrix. For example (1 2), (a b c) are row matrices of order 1 × 2 and 1 × 3 respectively.

(b) **Column Matrix :** A matrix of order m × 1 is called a column matrix. The matrices $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}$ are column matrices of order 3 × 1 and 2 × 1 respectively.

(c) **Zero matrix :** If all the elements of a matrix are zero it is called the zero matrix, (or null matrix) denoted by (0). The zero matrix may be of any order. Thus (0), (0, 0), $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ are all zero matrices.

(d) **Unit Matrix :** The square matrix whose elements on its main diagonal (left top to right bottom) are 1's and rest of its elements are 0's is called unit matrix. It is denoted by I and it may be of any order. Thus (1) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ are unit matrices of order 1, 2, 3 respectively.

(e) **Singular and non -singular matrices :** A square matrix A is said to be singular if and only if its determinant is zero and is said to be non-singular (or regular) if det A $\neq$ 0.

For example $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ is a non singular matrix.

For $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 4 - 6 = -2 \neq 0$ and $\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}$ is a singular matrix

i.e. $\begin{vmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{vmatrix} = 0$

Adjoint of a Matrix :

The adjoint of a matrix A is the transpose of the matrix obtained replacing each element a_{ij} in A by its cofactor A_{ij}. The adjoint of A is written as adj A. Thus if

$$A = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix} \quad \text{then adj } A = \begin{pmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & A_{33}\end{pmatrix}$$

Example – 1 : **Find inverse of the following matrices** $\begin{bmatrix}\mathbf{2} & \mathbf{-1}\\\mathbf{1} & \mathbf{3}\end{bmatrix}$

Sol^n : **(i)** Given $A = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$, |A| = 7

$A^{-1} = \dfrac{\text{adj } A}{|A|}$, $|A| \neq 0$

So it has inverse

Adj (A)

Minor of 2, $M_{11} = 3$, Cofactor of 2, $C_{11} = 3$

Minor of –1, $M_{12} = 1$, Cofactor of –1, $C_{12} = -1$

Minoir of 1, $M_{21} = -1$, Cofactor of 1, $C_{21} = 1$

Minor of 3, $C_{22} = 2$, Cofactor of 3, $C_{22} = 2$

$$\text{adj}(A) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}}{7} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

Assignment

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

   Calculate (i) AB (ii) BA

2. Find the inverse of the following : $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

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TRIGONOMETRY

COMPOUND ANGLES

INTRODUCTION :

The word Trigonometry is derived from Greek words ''Trigonos'' and metrons means measurement of angles in a triangle. This subject was originally devecpaed to solve geometric problems involving trigangles. The Hindu mathematicians Aryabhatta, Varahmira, Bramhaguptu and Bhaskar have lot of contaribution to trigonometry . Besides Hindu mathematicians ancient-Greek and Arwric mathematicians also contributed a lot to this subject. Trigonometry is used in many are as such as science of seismology, designing electrical circuits, analysing musical tones and studying the occurance of sun spots.

Trigonometric Functions :

Let θ be the meausre of any angle measured in radians in counter clockmise sense as show in Fig (1).

Let P(x, y) be any point an the terminal side of angle θ. The distance of P from O is OP = r = $\sqrt{x^2+y^2}$. the functions defined by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ ...(1) are called sine, cosine and tangent functions respectirely. These are called trigonometric functions. It followrs from (1) that $\sin^2\theta + \cos^2\theta = 1$. Other trigonomatric functions such as cosecant, secant and cotangent functions are defined as $\operatorname{cosec}\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$.

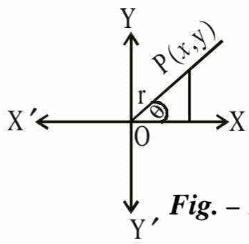
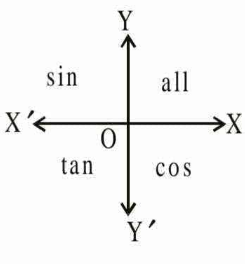


Fig. – 1

SIGN OF T-RATIOS :

The student may remember the signs of t-ratios in different quadrant with the help of the diagram



The sign of paricular t-ratio in any quadrant can be remembered by the word ''all-sin-tan-cos'' or ''add sugar to coffee''. What ever is written in a particular quadrant along with its reciprocal is +ve and the rest are negetive.

Table giving the values of trigonometrical Ratios of angles 0º, 30°, 45º, 60º & 90º

θ	0º	30º	45º	60º	90º
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

RELATED ANGLES :

Definitions : Two angles are said to be complementary angles if their sum is 90° and each angle is said to be the complement of the other.

Two angles are said to be supplementary if their sum is 180° and each angles is said to be the supplement of the other.

To Find the T-Ratios of angle (–θ) in terms of θ :

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. ***(Fig. 2)***

Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle P'OX$ will be taken as –θ. Join PP'. Let it meet OX at M.

Now Δ OPM $\equiv$ Δ P'OM, $\angle P'OM = -\theta$

OP' = OP, P'M = – PM

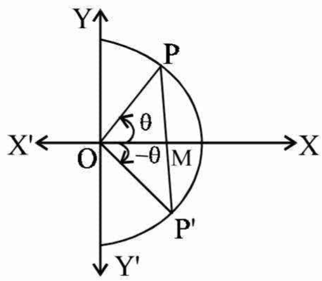


Fig. – 2

Now $\sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$

$\tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan\theta$

$\text{cosec}(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -\text{cosec}\,\theta$

$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta$

$\cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -\cot\theta$

To find the T-Ratios of angle (90° – θ) in terms of θ.

Let OPM be a right angled triangle with $\angle POM = 90°$, $\angle OMP = \theta$, $\angle OPM = 90° - \theta$. ***(Fig. 3)***

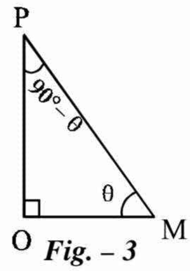


Fig. – 3

$\therefore \quad \sin(90° - \theta) = \frac{OM}{PM} = \cos\theta \quad \Rightarrow \quad \text{cosec}(90° - \theta) = \sec\theta$

$\cos(90° - \theta) = \frac{OP}{PM} = \sin\theta \quad \Rightarrow \quad \sec(90° - \theta) = \text{cosec}\,\theta$

$$\tan(90° - \theta) = \frac{OM}{OP} = \cot\theta \quad \Rightarrow \quad \cot(90° - \theta) = \tan\theta$$

To find the T-Ratios of angle (90° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. Draw PM and P'M' perpendiculars to the X-axis ***(Fig. 4)***

Now Δ POM ≅ Δ P'OM'

∴ P'M' = OM and OM' = –PM

$$\text{Now } \sin(90° + \theta) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90° + \theta) = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot\theta$$

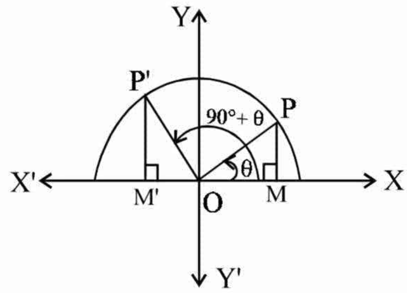


Fig. – 4

Similarly cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

and cot (90° + θ) = – tan θ

To Find the T-Ratios of angle (180° – θ) in terms of θ.

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle XOP = θ To obtain the angle 180° – θ let the radius vector start from OX and after revolving through 180° come to the position OX'. Let it revolve back through an angle θ in the clockwise direction and come to the position OP' so that the angle X'OP' is equal in magnitude but opposite in sign to the angle XOP. The angle XOP' is 180° – θ. ***(Fig.5)***

Draw P'M' and PM perpendicular to X'OX.

Now Δ POM ≅ Δ P'OM'.

∴ OM' = –OM and P'M' = PM

$$\text{Now } \sin(180° - \theta) = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin\theta$$

$$\cos(180° - \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan\theta$$

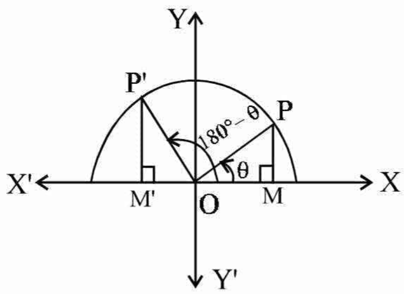


Fig. – 5

Similarly cosec (180° – θ) = cosec θ

sec (180° – θ) = –sec θ

and cot (180° – θ) = – cot θ

To Find the T-Ratios of angle (180° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. ***(Fig. 6)***

Now Δ POM ≅ Δ P'OM'.

∴ OM' = –OM and P'M' = –PM

$$\text{Now } \sin(180° + \theta) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan\theta$$

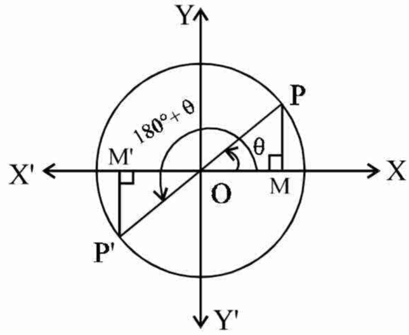


Fig. – 6

Similarly cosec (180° + θ) = cosec θ

sec (180° + θ) = –sec θ

and cot (180° + θ) = cot θ.

To Find the T-Ratios of angles (270° ± θ) in terms of θ.

The trigonometrical ratios of 270° – θ and 270° + θ in terms of those of θ, can be deduced from the above articles. For example

sin (270° – θ) = sin [180° + (90° – θ)]

= –sin (90° – θ) = – cos θ

cos (270° – θ) = cos [180° + (90° – θ)]

= – cos (90° – θ) = – sin θ

Similarly sin (270° + θ) = sin [180° + (90° + θ)]

= –sin (90° + θ) = – cos θ

cos (270° + θ) = cos [180° + (90° + θ)]

= – cos (90° + θ) = – (–sin θ) = sin θ

To Find the T-Ratios of angles (360° ± θ) in terms of θ.

We have seen that if n is any integer, the angle n. 360° ± θ is represented by the same position of the radius vector as the angle ± θ. Hence the trigonometrical ratios of 360° ±θ are the same as those of ±θ.

Thus sin (n. 360° + θ) = sin θ

cos (n. 360° + θ) = cos θ

sin (n. 360° – θ) = sin (–θ) = – sin θ

and cos (n. 360° – θ) = cos (–θ) = cos θ.

Examples :

cos (–720° – θ) = cos (–2 × 360° – θ) = cos (–θ) = cos θ

and tan (1440° + θ) = tan (4 × 360° + θ) = tan θ

In general when is any integer, n ∈ Z

(1) $\sin(n\pi + \theta) = (-1)^n \sin\theta$

(2) $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(3) $\tan(n\pi + \theta) = \tan\theta$ when n is odd integer

(4) $\sin\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$

(5) $\cos\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$

(6) $\tan\left(\frac{n\pi}{z} + \theta\right) = \cot\theta$

EVEN FUNCTION :

A function $f(x)$ is said to be an even function of x, if $f(x)$ satisfies the relation $f(-x) = f(x)$.

Ex. cosx, secx, and all even powers of x i.e, x^2, x^4, x^6...... are even function.

ODD FUNCTION :

A function $f(x)$ is said to be an odd function of x, if $f(x)$ satisfies the relation $f(-x) = -f(x)$.

Ex. sinx, cosec x, tan x, cot x and all odd powers of x i.e, x^3, x^5, x^7...... are odd function.

Example : ***Find the values of $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{-12}{13}$ and θ lies in the third quadrant.***

Solution : We have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

In third quadrant $\sin\theta$ is negetive, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = \frac{-5}{3}$$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-5}{13}\times\frac{13}{-12} = \frac{5}{12}$

Example : Find the values of

(i) tan (–900°) **(ii) sin 1230°**

Solution : (i) $\tan(-900°) = -\tan 900° = -\tan(10\times 90° + 0°) = -\tan 0° = 0$

(ii) $\sin(1230°) = \sin(6\times 180° + 150°) = \sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$

Example : Show that

$$\frac{\cos(90°+\theta).\sec(-\theta).\tan(180°-\theta)}{\sec(360°-\theta).\sin(180°+\theta).\cot(90°-\theta)} = -1 = \frac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$$

Solution : $\frac{\cos(90°+\theta).\sec(-\theta).\tan(180°-\theta)}{\sec(360°-\theta).\sin(180°+\theta).\cot(90°-\theta)} = \frac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$

ASSIGNMENT

1. Find the value of cos1°.cos2°..... cos 100°

2. Evaluale : $\tan\frac{\pi}{20}\cdot\tan\frac{3\pi}{20}\cdot\tan\frac{5\pi}{20}\cdot\tan\frac{7\pi}{20}\cdot\tan\frac{9\pi}{20}\cdot$

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COMPOUND, MULTIPLE AND SUB-MULTIPLE ANGLES

When an angle formed as the algebric sum of two or more angles is called a compound angles. Thus A + B and A + B + c are compound angles.

Addition Formulae

When an angle formed as the algebraical sum of two or more angles, it is called a compound angles. Thus A + B and A + B + C are compound angles.

Addition Formula :

(i) $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

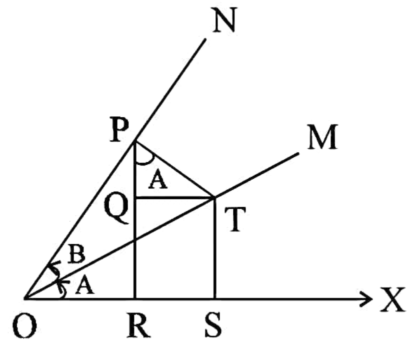

Fig. – 7

Proof : Let the revolving line OM starting from the line OX make an angle XOM = A and then further move to make.

$\angle MON = B$, so that $\angle XON = A + B$ *(Fig. 7)*

Let 'P' be any point on the line ON.

Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$

Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$

$\therefore$ We have from Δ OPR

(i) $\sin(A+B) = \dfrac{RP}{OP} = \dfrac{QR + PQ}{OP} = \dfrac{TS + PQ}{OP}$ $\quad (\because QR = TS)$

$= \dfrac{TS}{OP} + \dfrac{PQ}{OP} = \dfrac{TS}{OT} \cdot \dfrac{OT}{OP} + \dfrac{PQ}{PT} \cdot \dfrac{PT}{OP}$

$= \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A+B) = \dfrac{OR}{OP} = \dfrac{OS - RS}{OP} = \dfrac{OS}{OP} - \dfrac{RS}{OP}$

$= \dfrac{OS}{OP} - \dfrac{QT}{OP}$ $\quad [\because RS = QT]$

$= \dfrac{OS}{OT} \cdot \dfrac{OT}{OP} - \dfrac{QT}{PT} \cdot \dfrac{PT}{OP}$

$= \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B)}$

(dividing numerator and denominator by cos A cos B)

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

(iv) $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

[dividing of numerator and denominator by sin A sin B]

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - 1}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$

$\cot(A+B) = \dfrac{\cot A . \cot B - 1}{\cot B + \cot A}$

Cor : In the above formulae, replacing A by $\dfrac{\pi}{2}$ and B by x

We have

(i) $\sin\left(\dfrac{\pi}{2} + x\right) = \sin\dfrac{\pi}{2} . \cos x + \cos\dfrac{\pi}{2} . \sin x$

$= 1 . \cos x + 0 . \sin x = \cos x$

(ii) $\cos\left(\dfrac{\pi}{2} + x\right) = \cos\dfrac{\pi}{2} . \cos x - \sin\dfrac{\pi}{2} . \sin x$

$= 0 \times \cos x - 1 \times \sin x = -\sin x$

(iii) $\tan\left(\dfrac{\pi}{2} + x\right) = \dfrac{\sin\left(\dfrac{\pi}{2} + x\right)}{\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

(b) Difference Formulae :

(i) $\sin(A - B) = \sin A . \cos B - \cos A . \sin B$

(ii) $\cos(A - B) = \cos A . \cos B + \sin A . \sin B$

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

Proof : Let the reveolving line OM make an angle A with OX and then resolve back to make $\angle MON = B$ so that $\angle XON = A - B$. ***(Fig. 8)***

Let 'P' be any point on ON. Draw $PR \perp OX$,

$PT \perp OM$, $TS \perp OX$, $TQ \perp RP$ produced to Q.

Then $\angle TPQ = 90° - \angle PTQ = \angle QTM = A$

Now from Δ OPR, we have

(i) $\text{Sin}(A - B) = \dfrac{PR}{OP} = \dfrac{QR - QP}{OP} = \dfrac{TS - QP}{OP}$

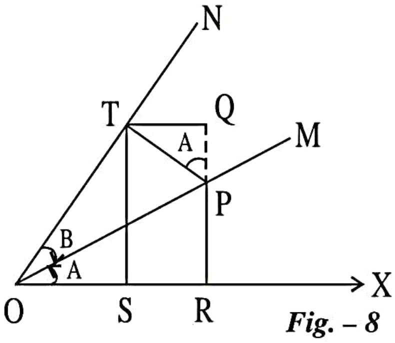

Fig. – 8

$$= \frac{TS}{OP} - \frac{QP}{OP}$$

$$= \frac{TS}{OT} \cdot \frac{OT}{OP} - \frac{OP}{PT} \cdot \frac{PT}{OP}$$

= sin A . cos B – cos A . sin B

(ii) $\cos(A - B) = \frac{OR}{OP} = \frac{OS + SR}{OP} = \frac{OS + TQ}{OP} = \frac{OS}{OP} + \frac{TQ}{QP}$

$$= \frac{OS}{OT} \cdot \frac{OT}{OP} + \frac{TQ}{PT} \cdot \frac{PT}{OP}$$

= cos A . cos B + sin A . sin B

(iii) $\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A.\cos B - \cos A.\sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Dividing the numerator and the denominator by cos A. cos B.

(iv) $\cot(A - B) = \frac{\cos(A - B)}{\sin(A - B)}$

$$= \frac{\cos A.\cos B + \sin A.\sin B}{\sin A.\cos B - \cos A.\sin B}$$

$$= \frac{\cot A.\cot B + 1}{\cot B - \cot A}$$

dividing the numerator and denominator by sin A. sin B

We can also deduce substraction formulae from addition formulae in the following manner.

sin(A – B) = sin[A + (–B)]

= sin A . cos (–B) + cos A . sin (–B)

= sin A . cos B + cos A . sin B

cos(A – B) = cos[A + (–B)]

= cos A . cos (–B) – sin A . sin (–B)

= cos A . cos B + sin A . sin B

$\tan(A - B) = \tan[A + (-B)] = \frac{\tan A + \tan(-B)}{1 - \tan A . \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

Example – 1 : **Find the value of tan 75° and hence prove that tan 75° + cot 75° = 4**

Solution: $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1 \times 1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$\therefore$ $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$$\therefore \quad \cot 75° = \frac{\sqrt{3}-1}{\sqrt{3}+1} \quad \left(\text{since } \cot\theta = \frac{1}{\tan\theta}\right)$$

$$\tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1} \qquad [\text{since } (a+b)(a-b) = a^2 - b^2]$$

$$\therefore \quad \tan 75° + \cot 75° = 4$$

Example – 2 : **If** $\sin A = \frac{1}{\sqrt{10}}$ **and** $\sin B = \frac{1}{\sqrt{5}}$ **show that** $A + B = \frac{\pi}{4}$

Solution: $\sin A = \frac{1}{\sqrt{10}}$

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{10-1}{10}} = \sqrt{\frac{9}{10}}$$

$$\therefore \cos A = \frac{3}{\sqrt{10}}$$

$$\sin B = \frac{1}{\sqrt{5}}, \cos B = \sqrt{1-\sin^2 B}$$

$$= \sqrt{1-\frac{1}{5}} = \sqrt{\frac{5-1}{5}} = \sqrt{\frac{4}{5}}$$

$$\therefore \cos B = \frac{2}{\sqrt{5}}$$

$$\text{Sin}(A + B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}}$$

$$= \frac{2+3}{\sqrt{50}} = \frac{2+3}{5\sqrt{2}}$$

$$\therefore \sin(A+B) = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sin(A+B) = \sin 45°$$

$$\therefore A + B = 45° = \frac{\pi}{4} \left[\text{since } 45° = \frac{180°}{4}\right]$$

Transformation of Sums or Difference in to Products

(a) We have that

$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$ ....(1)

$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$ ....(2)

$\cos(A + B) - \cos(A - B) = 2 \cos A \cos B$ ....(3)

$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$ ....(4)

Let $A + B = C$ and $A - B = D$

Then $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

Putting the value in formula (1), (2), (3) and (4) we get

$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (i)

$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$ .... (ii)

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (iii)

$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$ .... (iv)

for practice it is more convenient to quote the formulae verbally as follows :

Sum of two sines = 2 sin (half sum) cos (half difference)

Difference of two sines = 2 cos (half sum) sin (half difference)

Sum of two cosines = 2 cos (half sum) cos (half difference)

Difference of two cosines = 2 sin (half sum) sin (half difference reversed)

[The student should carefully notice that the second factor of the right hand member of IV is $\sin \frac{D-C}{2}$, not $\sin \frac{C-D}{2}$]

(b) To find the Trigonometrical ratios of Angle 2A in terms of those of A : sin 2A, cos 2A.

Since $\sin (A + B) = \sin A \cos B + \cos A \sin B$

putting $B = A$

$\sin (A + A) = \sin A \cos A + \cos A \sin A$

$\Rightarrow \sin 2A = 2 \sin A \cos A$ ..... (i)

$\cos (A + B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \cos (A + A) = \cos A \cos A - \sin A \sin A$

$\Rightarrow \cos 2A = \cos^2 A - \sin^2 A$ ..... (ii)

Also $\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A$ ..... (iii)

So $2 \sin^2 A = 1 - \cos 2A$ ..... (iv)

Also $\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$ ..... (v)

or $2 \cos^2 A = 1 + \cos 2A$ ..... (vi)

(c) Formula for tan 2A

since $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan 2A = \tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$

$= \frac{2 \tan A}{1 - \tan^2 A}$

Note this formula is not defined when $\tan^2 A = 1$ i.e, $\tan A = \pm 1$

(d) To express sin 2A and cos 2A in terms of tan A

$\sin 2A = 2 \sin A \cos A$

$$= 2 \frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2 A}} = \frac{2 \tan A}{\sec^2 A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(dividing numerator and denominator by $\cos^2 A$)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(e) To find the Trigonometrical formulae of 3A

$\sin 3A = \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos A \cdot \cos A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$

$= 3 \sin A - 4 \sin^3 A$

Again, $\cos 3A = \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A. \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A)$

$= 4 \cos^3 A - 3 \cos A$

Also $\tan 3A = \tan (2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$, provided $3 \tan^2 A \neq 1$ i.e, $\tan A \neq \pm \frac{1}{\sqrt{3}}$

(f) Submultiple Angles :

To express trigonometric ratios of A in terms of ratios of A/2

$\sin 2\theta = 2 \sin \theta \cos \theta$ (true for all value of θ)

Let $2\theta = A$ i.e. $\theta = \frac{A}{2}$

$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$ .... (i)

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

or $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$ ..... (ii)

$\cos A = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}$ ..... (iii)

Also, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \qquad \text{..... (iv)}$$

[Where $A \neq n\pi + \frac{\pi}{2}$, $(n \in I)$ and $A \neq (2n+1)\pi$]

Again, $\sin A = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

[dividing numerator and denomenator by $\cos^2\frac{A}{2}$)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

[where $A \neq (2n+1)\pi$, $n \in I$]

Similarly, $\cos A = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{1} = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

Now dividing numerator and denominator by $\cos^2\frac{A}{2}$

$\Rightarrow \cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$ [where $A \neq (2n+1)\pi$, $n \in I$].

Example –1 : Find the values of

(i) $\cos 22\frac{1}{2}^{o}$ (ii) $\sin 15^{o}$

Solution : **(i)** We have c $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$, putting $A = 45^{o}$

$$\cos 22\frac{1}{2}^{o} = \sqrt{\frac{1+\cos 45^{o}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) $\sin 15^{o} = \sin(45^{o} - 30^{o})$

$= \sin 45^{o}.\cos 30^{o} - \cos 45^{o}.\sin 30^{o}$

$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Example – 2: Prove that sinA.sin (60º – A).sin(60º + A) = $\frac{1}{4}$ sin3A

Solution : sinA.sin (60º – A) sin (60º + A)

= sin A. ($\sin^2 60^\circ - \sin^2 A$) [$\because$ sin (A + B). sin (A – B) = $\sin^2 A - \sin^2 B$]

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right] = \sin\left[\frac{3}{4} - \sin^2 A\right] = \frac{1}{4} \qquad [3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A$$

Example – 3: Prove that sin20º.sin40º.sin60º.sin80º = $\frac{3}{16}$

Solution : sin 60º. sin 20º.sin40º. sin 80º

$$= \frac{\sqrt{3}}{2}\,[\sin A.\ \sin(60^\circ - A).\ \sin(60^\circ + A)] \text{ where } A = 20^\circ$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{4}\cdot\sin 3A = \frac{\sqrt{3}}{8}\cdot\sin 60 = \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example – 4: If A + B + C = π and cos A = cos B . cos C show that tan B + tan C = tan A

Solution : L.H.S. = tan B + tan C

$$= \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin B.\cos C + \cos B.\sin C}{\cos B.\cos C}$$

$$= \frac{\sin(B+C)}{\cos B.\cos C} = \frac{\sin(\pi - A)}{\cos B.\cos C} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S. (Proved)}$$

Examples – 5: Prove the followings

(a) $\cot 7\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$

(b) $\tan 37\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$

Solution : **(a)** We know cot $\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$ (Choosing θ = 15)

$$= \cot 7\frac{1}{2}^\circ = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2+\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2\sqrt{6}+2\sqrt{2}+\sqrt{3}+\sqrt{3}+1+3}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$$

(b) We know $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ (Choosing θ = 75º)

$$\tan 37\frac{1^\circ}{2} = \frac{1-\cos 75^\circ}{\sin 75^\circ} = \frac{1-\cos(90^\circ-15^\circ)}{\sin(90^\circ-15^\circ)}$$

$$= \frac{1-\sin 15^\circ}{\cos 15^\circ} = \frac{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

Example – 6: If sin A = K sin B, prove that $\tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$

Solution : Given sin A = K sin B

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$$

Using componendo & dividendo

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{K+1}{K-1}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}.\sin\frac{A-B}{2}} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2}.\cot\frac{A-B}{2} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2} = \frac{K+1}{K-1}.\tan\frac{A-B}{2}$$

$$\Rightarrow \tan\frac{A-B}{2} = \frac{K-1}{K+1}\tan\frac{A+B}{2}$$

∴ L.H.S. = R.H.S. (Proved)

Example – 7: If $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$

Solution : $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$ (Given)

$$\tan^2\frac{\beta}{2} = \frac{1+e}{1-e}\tan^2\frac{\alpha}{2}$$

L.H.S = cos β

$$= \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}$$

$$= \frac{1 - e - \tan^2\frac{\alpha}{2} - e\tan^2\frac{\alpha}{2}}{1 - e + \tan^2\frac{\alpha}{2} + e\tan^2\frac{\alpha}{2}} = \frac{\left(1 - \tan^2\frac{\alpha}{2}\right) - e\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right) - e\left(1 - \tan^2\frac{\alpha}{2}\right)}$$

$$= \frac{\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}{\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}$$

$$= \frac{\cos\alpha - e}{1 - e\cos\alpha} = \text{R.H.S (Proved)}$$

Example – 8: If A + B + C = π, then Prove the following

(i) sin 2A + sin 2B + sin 2C = 4 sin A. sin B. sin C

Solution : L.H.S. = sin 2A + sin 2B + sin 2C

= 2 sin (A + B) . cos (A – B) + 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) + 2 sin C . cos C

= 2 sin C . cos (A – B) + 2 sin C. cos C

= 2 sin C [cos (A – B) + cos C]

= 2 sin C [cos (A – B) – cos (A + B)]

= 2 sin C . 2 sin A . sin B

= 4 sin A . sin B . sin C R.H.S. (Proved)

(ii) sin 2A + sin 2B – sin 2C = 4 cos A . cos B. sin C

Solution : L.H.S. = sin 2A + sin 2B – sin 2C

= 2 sin (A + B) . cos (A – B) – 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) – 2 sin C. cos C

= 2 sin C. cos (A – B) – 2 sin C. cos C

= 2 sin C [cos (A – B) – cos {π – (A + B)}]

= 2 sin C {cos (A – B) + cos (A + B)}

$$= 2\sin C\left\{2\cos\frac{A - B + A + B}{2}\,.\cos\frac{A - B - A - B}{2}\right\}$$

= 4 sin C. cos A. cos B.

(iii) $\mathbf{\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\ \sin\frac{B}{2}\ \cos\frac{C}{2}}$

Solution : L.H.S. = sin A + sin B – sin C

$$= 2\sin\frac{A + B}{2}\,.\cos\frac{A - B}{2} - 2\sin\frac{C}{2}\,.\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\,.\cos\frac{A - B}{2} - 2\sin\frac{C}{2}\,.\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{(-2)\sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right) . \sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right\}$$

$$= -4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\left(-\frac{B}{2}\right)$$

$$= 4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\frac{B}{2} = \text{R.H.S (Proved)}$$

ASSIGNMENT

1. If $\tan\alpha = \frac{1}{2}, \tan\beta = \frac{1}{3}$, then find the value of $(\alpha + \beta)$

2. Find the value of $\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$

3. Prove that $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$

4. If A + B = 45º, show that (1 + tanA) (1+ tanB) = 2

5. If $(1 - e)\ \tan^2\frac{\beta}{2} = (1+e)\ \tan^2\frac{\alpha}{2}$

Prove that $\cos\beta = \frac{\cos\alpha - e}{1 - e\cos\alpha}$

6. If A + B + C = π, prove that
cos2A + cos2B + cos 2C + 1 + 4 cos A.cosB.cosC = 0

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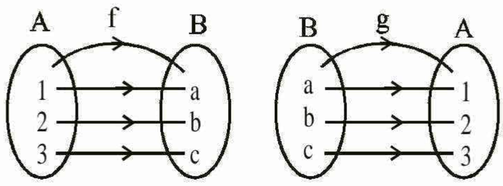
INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE FUNCTION :

If $f : A \rightarrow B$ be a bijective function or one to one onto function from set A to the set B. As the function is 1 – 1, every element of A is associated with a unique element of B. As the function is onto, there is no element of B which in not associated with any element of A. Now if we consider a function g from B to A, we have for $f \in B$ there is unique $x \in A$. This g is called inverse function of f and is denoted by f^{-1}.



INVERSE TRIGONOMETRIC FUNCTION :

We know the equation $x = \sin y$ means that y is the angle whose sine value is x then we have $y = \sin^{-1}x$ similarly $y = \cos^{-1}x$ if $x = \cos y$ and $y = \tan^{-1}x$ is $x = \tan y$ etc.

The function $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ are called inverse trigonometric function.

* Properties of inverse trigonometric function.

I. Self adjusting property :

(i) $\sin^{-1}(\sin\theta) = \theta$

(ii) $\cos^{-1}(\cos\theta) = \theta$

(iii) $\tan^{-1}(\tan\theta) = \theta$

Proof:

(i) Let $\sin\theta = x$, then $\theta = \sin^{-1}x$

$\therefore \sin^{-1}(\sin\theta) = \sin^{-1}x = \theta$

proofs of (ii) * (iii) as above.

II. Reciprocal Property :

(i) $\text{cosec}^{-1}\dfrac{1}{x} = \sin^{-1}x$

(ii) $\sec^{-1}\dfrac{1}{x} = \cos^{-1}x$

(iii) $\cot^{-1}\dfrac{1}{x} = \tan^{-1}x$

Proof :

(i) Let $x = \sin\theta$ then $\operatorname{cosec}\theta = \frac{1}{x}$

so that $\theta = \sin^{-1}x$ & $\theta = \operatorname{cosec}^{-1}\frac{1}{x}$

$\therefore\ \sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$

(ii) and (iii) may be proved similarly

III. Conversion property :

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x\ \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

Proof:

(i) Let $\theta = \sin^{-1}x$ so that $\sin\theta = x$

Now $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e., $\theta = \cos^{-1}\sqrt{1-x^2}$

Also $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Thus we have $\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Theorem – 1 : Prove that

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

Proof :

(i) Let $\sin^{-1}x = \theta$, then

$x = \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow\ \cos^{-1}x = \frac{\pi}{2}-\theta = \frac{\pi}{2} - \sin^{-1}x$

$\Rightarrow\ \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) and (iii) can be proved similarly.

Theorem – 2 : If xy < 1, then

$$\mathbf{\tan^{-1}x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

Proof : Let $\tan^{-1}x = \theta_1$ and $\tan^{-1} y = \theta_2$

Then

$\tan \theta_1 = x$ and $\tan \theta_2 = y$

$$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Theorem – 3 : $\mathbf{\tan^{-1}x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)}$

Proof : Let $\tan^{-1}x = \theta$, and $\tan^{-1}y = \theta_2$

$$\Rightarrow \tan \theta_1 = x \text{ and } \tan \theta_2 = y$$

$$\Rightarrow \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1, \tan\theta_2} = \frac{x-y}{1+xy}$$

$$\Rightarrow \theta_1 - \theta_2 = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

Note : $\tan^{-1} + \tan^{-1} y + \tan^{-1}z$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Theorem – 4 : Prove that :

(i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

Proof :

(i) Let $\sin^{-1}x = \theta$, Then, $x = \sin\theta$

$$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta.\ \sqrt{1-\sin^2\theta}$$

$$= 2x\sqrt{1-x^2}$$

$$\Rightarrow 2\theta = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \quad \Rightarrow 2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$$

(ii) Let $\cos^{-1}x = \theta$ Then, $x = \cos\theta$

$$\therefore \cos 2\theta = (2\cos^2\theta - 1) = 2x^2 - 1$$

$$\Rightarrow 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$$

Theorem – 5 : Prove that

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$

(iii) $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$

Proof :

(i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$, Then

$\sin \theta_1 = x$ and $\sin \theta_2 = y$

$\therefore \quad \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

$= \sin\theta_1 \sqrt{1-\sin^2\theta_2} + \sqrt{(1-\sin^2\theta_1)}\ \sin\theta_2$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad \theta_1 + \theta_2 = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}\ ]$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}\ ]$

The other results may be proved similarly.

Example – 1 : **If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$**

Solution : Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}\ ) = (\pi - \cos^{-1}z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi - \cos^{-1}z)$

$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos(\cos^{-1}z) = -z$

$\Rightarrow \quad xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2) = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$ (Proved)

Example – 2 : **Find the value of $\cos\tan^{-1}\cot\cos^{-1}\dfrac{\sqrt{3}}{2}$**

Solution : $\cos^{-1}\dfrac{\sqrt{3}}{2} = \theta \Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{6} \Rightarrow \cos^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6}$

$\therefore \quad \cos\tan^{-1}\cot\cos^{-1}\dfrac{\sqrt{3}}{2} = \cos\tan^{-1}\cot\dfrac{\pi}{6}$

$= \cos\tan^{-1}\sqrt{3}\left[\because\ \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}\right] = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

Example – 3 : Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

Solution : L.H.S $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} \qquad \left(\because 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}\right)$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{1}{2}+\frac{1}{7}}{1-\frac{1}{2}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{9}{14}}{\frac{13}{14}} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{9}{13}$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{9}{13}}{1-\frac{1}{2}\times\frac{9}{13}} = \tan^{-1}\frac{\frac{31}{26}}{\frac{17}{26}} = \tan^{-1}\frac{31}{17} = \text{R.H.S. (Proved)}$$

Example – 4 : Prove that $\cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

Solution : L.H.S. $= \cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \qquad \left[\because \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \tan^{-1}\frac{4}{5}\right]$$

$$= \tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}.\frac{4}{5}} = \tan^{-1}\frac{\frac{5+36}{45}}{\frac{45-4}{45}} = \tan^{-1}\frac{\frac{41}{45}}{\frac{41}{45}}$$

$$= \tan^{-1}1 = \frac{\pi}{4} \text{ R.H.S. (Proved)}$$

ASSIGNMENT

1. Find the value of $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

2. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$. Show that

$$x.\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

3. If s $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$. Find the value of x.

❒ ☸ ❒

CO-ORDINATE GEOMETRY

STRAIGHT LINE

CO-ORDINATE SYSTEM

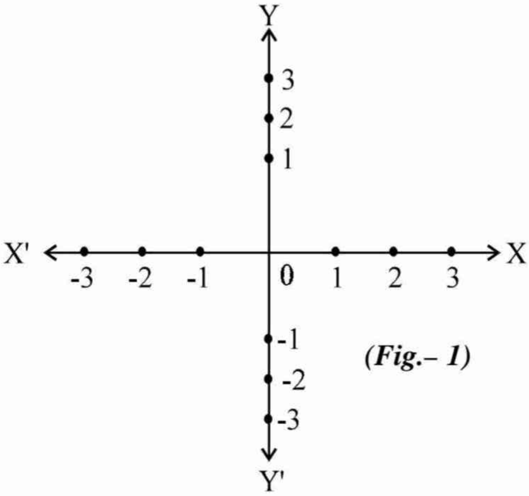
We represent each point in a plane by means of an ordered pair of real numbers, called co-ordinates. The branch of mathematics in which geometrical problems are solved through algebra by using the co-ordinate system, is known as co-ordinate geometry or analytical geometry.

Rectangular co-ordinate Axes

Let X'OX and YOY' be two mutually perpendicular lines (called co-ordinate axes), intersecting at the point O. ***(Fig.1)***.We call the point O, the origin, the horizontal line X'OX, the x-axis and the vertical line YOY', the y-axis.

We fix up a convenient unit of length and starting from the origin as zero, mark. distances on x-axis as well as y-axis.

The distance measured along OX and OY are taken as positive while those along OX' and OY' are considered negative.



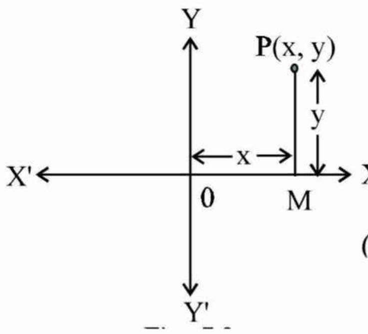
(Fig.– 1)

Cartesian co-ordinates of a point

Let X'OX and YOY' be the co-ordinate axes and let P be a point in the Euclidean plane ***(Fig.2)***. From P draw PM ⊥ X'OX.

Let OM = x and PM = y, Then the ordered pair (x, y) represents the cartesian co-ordinates of P and we denote the point by P(x, y). The number x is called the x-co-ordinate or abscissa of the point P, while y is known as its y-co-ordinate or ordinate.

Thus, for a given point the abscissa and the ordinate are the distances of the given point from y- axis and x-axis respectively.



(Fig.– 2)

Quadrants

The co-ordinate axes X'OX and Y'OY divide the plane in to four regions, called quadrants.

The regions XOY, YOX', X'OY' and Y'OX are known as the first, the second, the third and the fourth quadrant respectively. ***(Fig.3)***

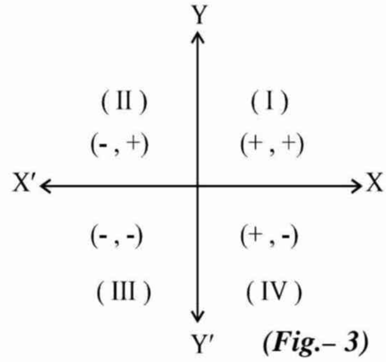
In accordance with the convention of signs defined above for a point (x, y) in different quadrants we have

1st quadrant : $x > 0$ and $y > 0$

2nd quadrant : $x < 0$ and $y > 0$

3rd quadrant : $x < 0$ and $y < 0$

4th quadrant : $x > 0$ and $y < 0$



(Fig.– 3)

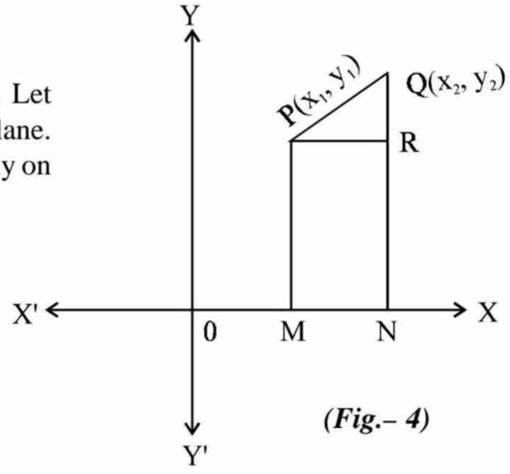
DISTANCE BETWEEN TWO GIVEN POINTS

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

Proof : Let X'OX and YOY' be the co-ordinate axes ***(Fig.4)***. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points in the plane. From P and Q draw perpendicular PM and QN respectively on the x-axis. Also draw PR $\perp$ QN.



(Fig.– 4)

Then, $OM = x_1$, $ON = x_2$

$PM = y_1$ & $QN = y_2$

$\therefore$ $PR = MN = ON - OM = x_2 - x_1$

and $QR = QN - RN = QN - PM = y_2 - y_1$

Now from right angled triangle PQR,

we have $PQ^2 = PR^2 + QR^2$ [by Pythagoras theorem]

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

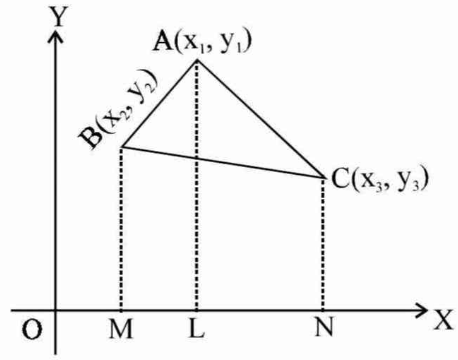
$$\therefore |PQ| = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

Cor : The distance of a point P(x, y) from the origin O (0, 0) is

$$= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Area of a triangle :

Let ABC be a given triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. From the vertices A, B and C draw perpendiculars AL, BM and CN respectively on x-axis. ***(Fig.5)***.



(Fig.– 5)

Then, $ML = x_1 - x_2$; $LN = x_3 - x_1$ and $MN = x_3 - x_2$

$\therefore$ Area of Δ ABC

= area of trapezium ALMB + area of trapezium ALNC – area of trapezium BMNC

$$= \frac{1}{2}(AL + BM) . ML + \frac{1}{2}(AL + CN) . LN - \frac{1}{2}(MB + CN) . MN$$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 + x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 - x_3y_2 - x_3y_3 + x_2y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

In determinant form, we may write

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for collinearity of Three points :

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are colliner, i.e. lie on the same straight line, if the area of ΔABC is zero. So the required condition for A, B, C to be collinear is that

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

$$\Rightarrow \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Formula for Internal Divisions :

The co-ordinates of a point P which divides the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio m : n are given by

$$\bar{x} = \frac{mx_2 + nx_1}{m+n}, \quad \bar{y} = \frac{my_2 + ny_1}{m+n}$$

Example – 1 : **In what ratio does the point (3, –2) divide the line segment joining the points (1, 4) and (–3, 16) :**

Solution : Let the point C (3, –2) divide the segment joining A(1, 4) and B (–3, 16) in the ratio K: 1

The co-ordinates of 'C' are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But we are given that the point C is (3, –2)

$\therefore$ We have $\frac{-3k+1}{k+1} = 3$

or $-3k + 1 = 3k + 3$

or $-6k = 2$

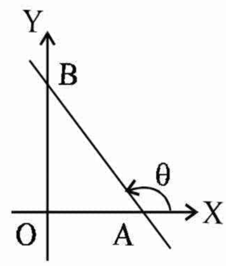
$\therefore$ $k = -\frac{1}{3}$

$\therefore$ C divides AB in the ratio 1 : 3 externally.

SLOPE OF A LINE

Angle of Inclination : The angle of inclination or simply the inclination of a line is the angle θ made by the line with positive direction of x-axis, measured from it in anticlock wise direction ***(Fig. 6)***.

Slope or gradient of a line : If θ is the inclination of a line, then the value of $\tan\theta$ is called the slope of the line and is denoted by m.



(Fig.– 6)

CONDITIONS OF PARALLELISM AND PERPENDICULARITY

1. Two lines are parallel if and only if their slopes are equal.

2. Two lines with slope m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$

3. The slope of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

4. The equation of a line with slope m and making an intercept 'c' on y-axis is given by y = mx + c.

Proof : Let AB be the given line with inclination θ so that $\tan\theta = m$. Let it intersect the y-axis at C so that OC = c. *(Fig.7)*

Let it intersect the x-axis at A.

Let P(x, y) be any point on the line.

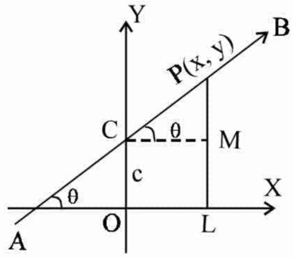
Draw PL perpendicular to x-axis and $CM \perp PL$

Clearly, $\angle MCP = \angle OAC = \theta$

CM = OL = x ;

and PM = PL – ML = PL – OC = y – c

Now, from rt. angled Δ PMC



(Fig.– 7)

We get $\tan\theta = \dfrac{PM}{CM}$ or $m = \dfrac{y-c}{x}$

or $y = mx + c$, which is required equation of the line.

5. The equation of a line with slope m and passing through a point (x_1, y_1) is given by $(y - y_1) = m(x - x_1)$

6. The equation of a line through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

7. The equation of a straight line which makes intercepts of length 'a' and 'b' on x-axis and y-axis respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Proof : Let AB be a given line meeting the x-axis and y-axis at A and B respectively *(Fig.8)*.

Let OA = a and OB = b

Then the co-ordinates of A, B are A(a, 0) and B(0, b)

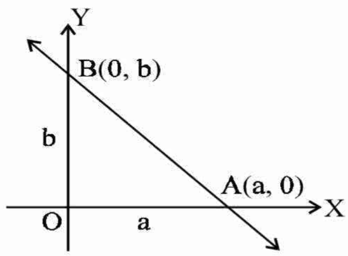
$\therefore$ The equation of the line joining A & B is

$$(y - 0) = \frac{b-0}{0-a}(x - a)$$

$$\Rightarrow y = \frac{-b}{a}(x - a)$$

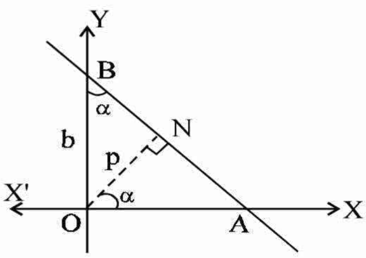
$$\Rightarrow \frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$



(Fig.– 8)

8. Let P be the length of perpendicular from the origin to a given line and α be the angle made by this perpendicular with the positive direction of x-axis. Then the equation of the line is given by $x\cos\alpha + y\sin\alpha = P$



(Fig.– 9)

Conditions for two lines to be coincident, parallel, perpendicular or Intersect :

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) conicident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Perpendicular, if $a_1a_2 + b_1b_2 = 0$;

(iv) Intersecting, if they are neither coincident nor parallel.

Example – 1 : **Find the equation of the line which passes through the point (3, 4) and the sum of its intercept on the axes is 14.**

Sol^n : Let the intercept made by the line on x-axis be 'a' and 'y'- axis be 'b'

i.e. $a + b = 14$ i.e, $b = 14 - a$

$\therefore$ Equation of the line is given by

$$\frac{x}{a} + \frac{y}{14-a} = 1 \text{ .... (i)}$$

As the point (3, 4) lies on it, we have

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

or $3(14 - a) + 4a = 14a - a^2$

or $42 - 3a + 4a = 14a - a^2$

or $a^2 - 13a + 42 = 0$

or $(a - 7)(a - 6) = 0$

or $a = 7$ or $a = 6$

Putting these values of a in (i)

$$\frac{x}{7} + \frac{y}{7} = 1 \quad \text{or} \quad x + y = 7$$

and $\frac{x}{6} + \frac{y}{8} = 1$ or $4x + 3y = 24$

Example – 2 : **Find the equation of the line passing through (–4, 2) and parallel to the line $4x - 3y = 0$**

Sol^n : Any line passing thorugh (–4, 2) whose equation is given by

$(y - 2) = m(x + 4)$ .... (i)

and parallel to the given line $4x - 3y = 0$

whose slope is $y = \frac{4}{3}x$

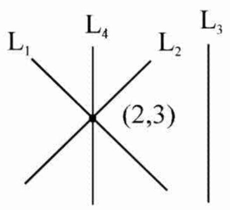
Here 'm' $= \frac{4}{3}$

It's equation is

$$(y - 2) = \frac{4}{3}(x + 4)$$

$3y - 6 = 4x + 16$

or $4x - 3y + 22 = 0$



(Fig.– 10)

Example – 3 : **Find the equation of the line passing through the intersection of $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$ and parallel to the line $x + y - 2 = 0$**

Sol^n : Point of intersection of $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$

$$\left(\frac{-1\times 6-(-4)(-1)}{2(-4)-3(-1)}, \frac{(-1)\times 3-6(2)}{2(-4)-3(-1)}\right)$$

$$= \left(\frac{-6-4}{-8+3}, \frac{-3-12}{-8+3}\right) = \left(\frac{-10}{-5}, \frac{-15}{-5}\right) = (2, 3)$$

Any line parallel to the line $x + y - 2$ is given by $x + y + k = 0$.... (i)

Since the line passes through (2, 3) hence it satisfies the equation (i)

So, $2 + 3 + k = 0$

$\Rightarrow k = -5$

Now putting the value of k in equation (i), we get $x + y - 5 = 0$

$\therefore$ Equation of the line is $x + y - 5 = 0$

Assignment

1. Find the equation of a line parallel to $2x + 4y - 9 = 0$ and passing through the point $(-2, 4)$
2. Find the co-ordinates of the foot of the perpendicular from the point (2, 3) on the line $3x - 4y + 7 = 0$
3. Find the equation of the line through the point of intersection of $3x + 4y - 7 = 0$ and $x - y + 2 = 0$ and which is parallel to the line $5x - y + 11 = 0$

❐ ☸ ❐

CIRCLE

A circle is the locus of a point which moves in a plane in such a way that it's distance from a fixed point is always constant.

The fixed point is called the centre of the circle and the constant distance is called its radius.

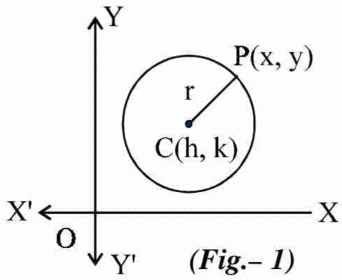
Equation of a circle (Standard form)

Let C(h, k) be the centre of a circle with radius 'r' and let P (x, y) be any point on the circle ***(Fig.1)***.

Then $CP = r \Rightarrow CP^2 = r^2$

$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$

Which is required equation of the circle.



(Fig.– 1)

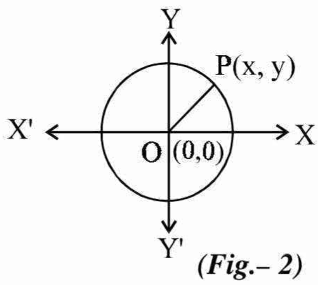
Cor. The equation of a circle with the centre at the origin and radius r, is $x^2 + y^2 = r^2$ ***(Fig.2)***.

Proof : Let O (0, 0) be the centre and r be the radius of a circle and let P (x, y) be any point on the circle.

Then $OP = r \Rightarrow OP^2 = r^2$

$\Rightarrow (x - 0)^2 + (y - 0)^2 = r^2$

$\Rightarrow x^2 + y^2 = r^2$



(Fig.– 2)

Example – 1. **Find the equation of a circle with centre (–3, 2) and radius 7.**

Solⁿ **:** The required equation of the circle is

$[x - (-3)]^2 + (y - 2)^2 = 7^2$

or $(x + 3)^2 + (y - 2)^2 = 49$

or $x^2 + y^2 + 6x - 4y - 36 = 0$

Example – 2. **Find the equation of a circle whose centre is (2, –1) and which passes through (3, 6)**

Solⁿ **:** Since the point P (3, 6) lies on the circle, its distance from the centre C (2, –1) is therefore equal to the radius of the circle.

$\therefore$ Radius = CP = $\sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}$

So, the required equation of the circle is

$(x - 2)^2 + (y + 1)^2 = 50$

or $x^2 + y^2 - 4x + 2y - 45 = 0$

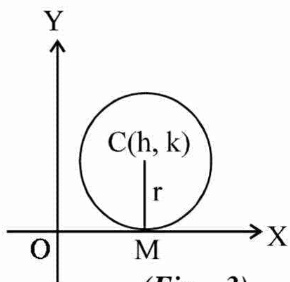
Example – 3. **Find the equation of a circle with centre (h, k) and touching the x-axis *(Fig.3)*.**

Solⁿ **:** Clearly, the radius of the circle = CM = r = k

So, the required equation

$(x - h)^2 + (y - k)^2 = k^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$



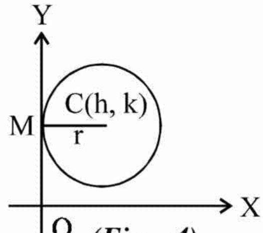
(Fig.– 3)

***Example – 4*. Find the equation of a circle with centre (h,k) and touching y-axis*(Fig.4)*.**

***Sol**ⁿ* **:** Clearly, the radius of the circle = CM = r = h

So, the required equation is $(x - h)^2 + (y - k)^2 = h^2$

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$



(Fig.– 4)

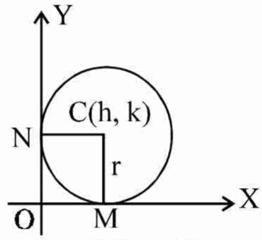
***Example – 5*. Find the equation of a circle with centre (h,k) and touching both the axes *(Fig.5)*.**

***Sol**ⁿ* **:** Clearly, radius, CM = CN = r

i.e. h = k = r (say)

∴ the equation of the circle is $(x - r)^2 + (y - r)^2 = r^2$

or $x^2 + y^2 - 2r(x + y) + r^2 = 0$



(Fig.– 5)

GENERAL EQUATION OF A CIRCLE

Theorem : The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$
And, every such equation represents a circle.

Proof : The standard equation of a circle with centre (h, k) and radius r is given by

$(x - h)^2 + (y - k)^2 = r^2$

Or $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

This is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

Where $h = -g, k = -f$ and $c = (h^2 + k^2 - r^2)$

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given condition.

Then, $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 - c)$

$\Rightarrow (x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow [x - (-g)]^2 + [y - (-f)]^2 = \left[\sqrt{g^2 + f^2 - c}\right]^2$

$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$

Where $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre (–g, –f) and radius.

$= \sqrt{g^2 + f^2 - c}$, provided $g^2 + f^2 > c$.

EQUATION OF A CIRCLE WITH GIVEN END POINTS OF A DIAMETER

Theorem : The equation of a circle described on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_1)$ as a diameter, is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

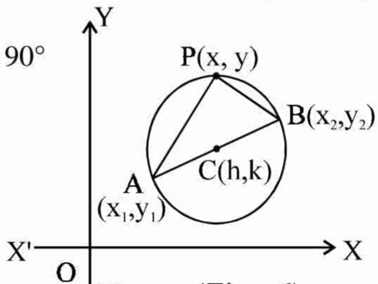
Proof : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end point of a diameter of the given circle and let P (x, y) be any point on the circle *(**Fig.6**)*.

Since the angle in a semi-circle is a right angle, we have ∠APB = 90°

Now slope of AP = $\left[\dfrac{y - y_1}{x - x_1}\right]$

And, slope of BP = $\left(\dfrac{y - y_2}{x - x_2}\right)$

Since AP ⊥ BP, we have



(Fig.– 6)

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_1}\right)=-1$$

Or $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

***Example – 1*. Find the equation of a circle whose end points of diameter are (3, 4) and (–3, –4)**

Sol^n^. : The required equation of the circle is $(x-3)(x+3)+(y-4)(y+4)=0$

i.e. $x^2-9+y^2-16=0$

or $x^2+y^2=25$

***Example – 2*. Find the centre and radius of the circle.**

$$\mathbf{x^2+y^2-6x+4y-36=0}$$

***Sol^n^*. :** Comparing the equation with

$x^2+y^2+2gx+2fy+c=0$

We get $2g=-6$, $2f=4$ and $c=-36$

or $g=-3$, $f=2$ and $c=-36$

$\therefore$ Centre of the circle is $(-g,-f)$, i.e. $(3,-2)$

And radius of the circle.

$=\sqrt{g^2+f^2-c}=\sqrt{9+4+36}=7$

Assignment

1. Find the centre and radius of each of the following circles
   $x^2+y^2+x-y-4=0$
2. Find the equation of the circle whose centre is (–2, 3) and passing through origin
4. Find the equation of the circle having centre at (1, 4) and passing through (–2, 1).
4. Find the equation of the circle passing through the points (1, 3) (2, –1) and (–1, 1).

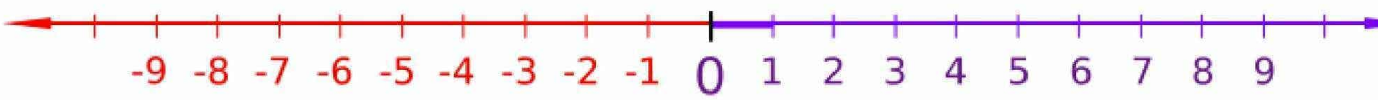
❒ ☸ ❒

Co-Ordinate System

In geometry, a **coordinate system** is a system which uses one or more numbers, or **coordinates**, to uniquely determine the position of a point. The order of the coordinates is significant and they are sometimes identified by their position in an ordered tuple and sometimes by a letter, as in "the *x* coordinate".

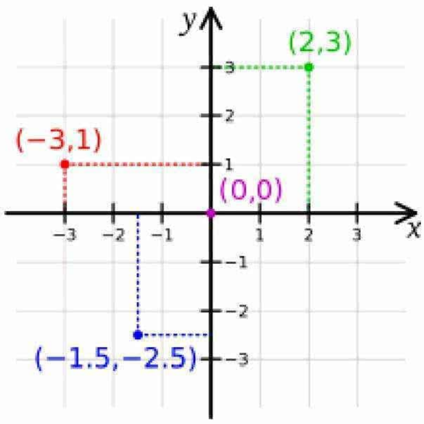
Number Line

The simplest example of a coordinate system is the identification of points on a line with real numbers using the *number line*. In this system, an arbitrary point *O* (the*origin*) is chosen on a given line. The coordinate of a point *P* is defined as the signed distance from *O* to *P*, where the signed distance is the distance taken as positive or negative depending on which side of the line *P* lies. Each point is given a unique coordinate and each real number is the coordinate of a unique point.[4]



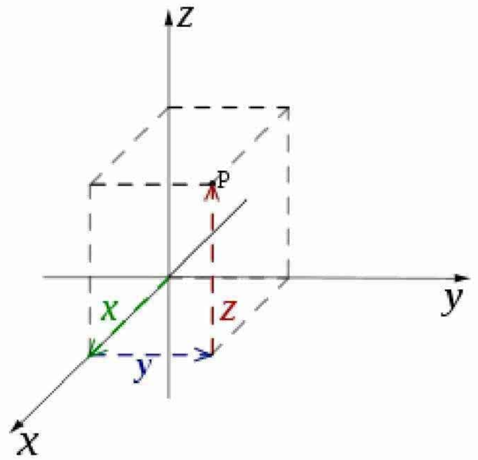
Cartesian Co-ordinate System

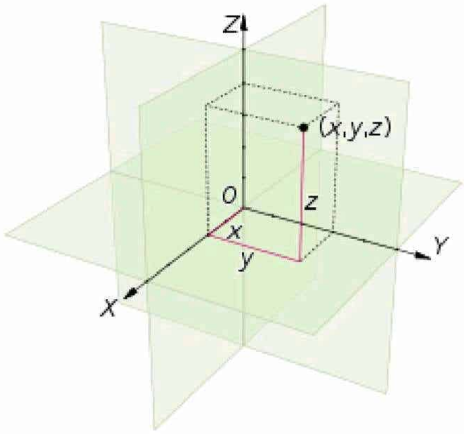
In the plane, two perpendicular lines are chosen and the coordinates of a point are taken to be the signed distances to the lines.



Three Dimension

In three dimensions, three perpendicular planes are chosen and the three coordinates of a point are the signed distances to each of the planes.





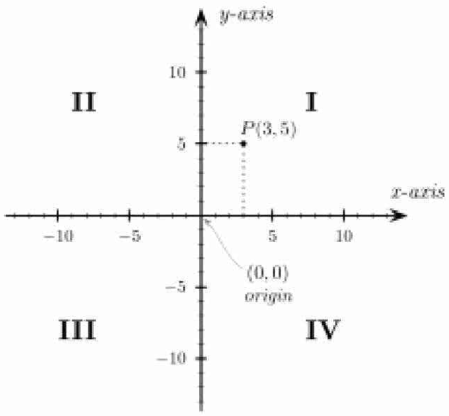
Choosing a Cartesian coordinate system for a three-dimensional space means choosing an ordered triplet of lines (axes) that are pair-wise perpendicular, have a single unit of length for all three axes and have an orientation for each axis. As in the two-dimensional case, each axis becomes a number line. The coordinates of a point *P* are obtained by drawing a line through *P* perpendicular to each coordinate axis, and reading the points where these lines meet the axes as three numbers of these number lines.

Alternatively, the coordinates of a point *P* can also be taken as the (signed) distances from *P* to the three planes defined by the three axes. If the axes are named *x*, *y*, and *z*, then the *x*-coordinate is the distance from the plane defined by the *y*and *z* axes. The distance is to be taken with the + or − sign, depending on which of the two half-spaces separated by that plane contains *P*. The *y* and *z* coordinates can be obtained in the same way from the *x*–*z* and *x*–*y* planes respectively.

The Cartesian coordinates of a point are usually written in parentheses and separated by commas, as in (10, 5) or(3, 5, 7). The origin is often labelled with the capital letter *O*. In analytic geometry, unknown or generic coordinates are often denoted by the letters *x* and *y* on the plane, and *x*, *y*, and *z* in three-dimensional space.

The axes of a two-dimensional Cartesian system divide the plane into four infinite regions, called **quadrants**, each bounded by two half-axes.

Similarly, a three-dimensional Cartesian system defines a division of space into eight regions or **octants**, according to the signs of the coordinates of the points. The convention used for naming a specific octant is to list its signs, e.g. (+ + +) or (− + −).



Distance between two points

The distance between two points of the plane with Cartesian coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is the Cartesian version of Pythagoras' theorem. In three-dimensional space, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

which can be obtained by two consecutive applications of Pythagoras' theorem.

Example :

Prive that the point A(-1,6,6),B(-4,9,6),C(0,7,10) form the vertices of a right angled tringled.

Solution :

By distance formula

$AB^2 = (-4+1)^2 + (9-6)^2 + (6-6)^2 = 9 + 9 = 18$

$BC^2 = (0+4)^2 + (7-9)^2 + (10-6)^2 = 16 + 4 + 16 = 36$

$AC^2 = (0+1)^2 + (7-6)^2 + (10-6)^2 = 1 + 1 + 16 = 18$

Which gives $AB^2 + AC^2 = 18 + 18 = 36$

Hence ABC is a right angled isosceles triangle

Derivation Of Distance Formula

Fig

The distance between the point P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is given by

PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Proof:

Let $\overline{P`Q`}$ be the projection of $\overline{PQ}$ on the XY plane. $\overline{PP`}$ and . $\overline{QQ`}$ are parallel. So $\overline{PP`}$ and $\overline{QQ`}$ are co-planar. And $\overline{PP`QQ`}$ is a plane quadrilateral.

Let R be a point on $\overleftrightarrow{QQ`}$ so that $\overline{PR}$ || $\overline{P`Q`}$.

Since $\overleftrightarrow{P`Q`}$ lies on the XY plane and $\overleftrightarrow{PP`}$ is perpendicular to this plane , it follow from the definition of perpendicular geometry to a plane that $\overline{PP`}$ perpendicular $\overline{P`Q`}$. Similarly $\overline{QQ`}$ perpendicular to $\overline{PP`}$. $\overline{PR}$ being parallel to $\overline{P`Q`}$. It follow from plane geometry that PP`Q`R is a rectangle. So PR = P`Q`. and

< PRQ is a right angle.

P\` and Q\` being the projection of point P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) on the XY plane, they are given by P\`$(x_1, y_1, 0)$ and Q$(x_2, y_2, 0)$. Therefore by the distance formula in the geometry of R^2.

P\`Q\` = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

In the rectangle PP\`Q\`R

P\`P = Q\`R

Therefore QR = | $z_2 - z_1$ |

In the right angled tringle PRQ , $PQ^2 = PR^2 + RQ^2$

= $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Derive the division formula

Fig

If R(x,y,z) divides the segment joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in ratio m : n ie $\frac{PR}{QR} = \frac{m}{n}$,then x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

Proof :

Let P\`, Q\` and R\` be the feet of the perpendicular from P, Q, R on the xy plane. Being perpendicular on the same plane $\overleftrightarrow{PP`}$ $\overleftrightarrow{QQ`}$ $\overleftrightarrow{RR`}$ are parallel lines. Since these parallel lines have a common transversal $\overleftrightarrow{PQ}$ they are co-planar. Let M and N be points on $\overleftrightarrow{RR`}$ and $\overleftrightarrow{QQ`}$ such that $\overleftrightarrow{PM}$

perpendicular $\overleftrightarrow{RR`}$ and $\overleftrightarrow{RN}$ perpendicular $\overleftrightarrow{QQ`}$. Since P`,R` and Q` are common to the xy-plane and plane of $\overleftrightarrow{PP`}\ \overleftrightarrow{QQ`}\ \overleftrightarrow{RR`}$ they must collinear because two plane intersect along a line.

It follows from the definition of the perpendicular to a plane that $< PP`R`, < RR`Q`\ and\ < QQ`R`$ are all right angles. It now follows from plane geometry that PP`R`M and RR`Q`N are rectangles. Also triangles RPM and QRN are similar

Hence $\frac{m}{n} = \frac{PR}{RQ} = \frac{PM}{RN} = \frac{P`R`}{R`Q`}$

$(\therefore PM = P`R`\ and\ RN = R`Q`\ in\ the\ corresponding\ rectangle)$

Thus the point R` divides the segment $\overline{P`Q`}$ internally in the ration m : n.

P`, R` and Q` being projection of P(x_1, y_1, z_1) , R(x,y,z) and Q(x_2, y_2, z_2) on the xy plane have co-ordinate respectively $(x_1, y_1, 0)$,(x,y,0), $(x_2, y_2, 0)$.

If we restrict our consideration to the xy plane only we can regard the point P`,R`,Q` as having coordinate (x_1, y_1) ,(x,y), (x_2, y_2) .

Thus on the xy-plane the point R`(x,y) divides the segment joining P(x_1, y_1) and Q(x_2, y_2) internally in the ratio given by x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$

Simillarly considering projection of P,Q,R on on another co-ordinate plane say YZ plane we can prove y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

Thus we have x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

External Division Formula

If R(x,y,z) devides the segment $\overline{PQ}$ joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) externally in ratio m : n ie $\frac{PR}{QR} = \frac{m}{n}$ then x = $\frac{mx_2-nx_1}{m-n}$, y = $\frac{my_2-ny_1}{m-n}$ and z = $\frac{mz_2-nz_1}{m-n}$

Example :

Find the ratio in which the line segment joining points (4,3,2) and (1,2,-3) is divided by the co-ordinate planes.

Solution :

Let the given points be denoted by A(4,3,2) and B(1,2,-3). If Q is the point where the line through A and B is met by xy-plane, then the co-ordinate of Q are $(\frac{k+4}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+2}{k+1})$, since Q devides $\overline{AB}$ in a ratio k:1 for some real value k. But being a point on the xy-plane, its z co-ordinate is zero.

Hence $\frac{-3k+2}{k+1} = 0 \ or \ k = \frac{2}{3}$

Similarly $\overline{AB}$ meets the xy-plane has its y-co-ordinate zero. Hence equating the y-co-ordinate to zero we get

$\frac{2k+3}{k+1} = 0 \ \ or \ k = -\frac{3}{2}$ ie the xz plane devides in a ratio 3:2. Equating x co-ordinate to zero we get $\frac{k+4}{k+1}$

k = -4.

Ie yz-plane divides $\overline{AB}$ externally in a ratio 4:1.

Direction Cosine and Direction Ratio

Fig

Let L be a line in space. Consider a ray R parallel to L with vortex at origin. (R can be taken as either $\overrightarrow{OP}$ or $\overrightarrow{OP`}$). let α, β, γ be the inclination between the ray R and $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ respectively.Them we define the direction cosine of L as $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Usually direction cosine of a line are denoted as <l,m,n>. for the above line l = $\cos\alpha$, m = $\cos\beta$,

n = $\cos\gamma$.

In the definition of the direction cosine of L the ray can be either $\overrightarrow{OP}$ or $\overrightarrow{OP`}$. Therefore if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosine of L then $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$, $\cos(\pi - \gamma)$ can also be considered as direction cosine of L . The two set of direction cosine corresponds to the two opposite direction of a line L.

The direction cosine of the ray $\underset{OP}{\rightarrow}$ are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and of the ray $\underset{OP`}{\rightarrow}$ are $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$, $\cos(\pi-\gamma)$.

Property Of Direction Cosine

A. Let O bethe origin and direction cosine of $\underset{OP}{\rightarrow}$ be l,m,n. If OP = r and P has a co-ordinate (x,y,z) then

   x = lr, y = mr, z = nr.

B. If l,m,n are direction cosines of a line then
   $l^2+m^2+n^2=1$

Direction Ratio

Let l,m,n be the direction cosine of the line such that none of the direction cosine is zero.

If a,b,c are non zero real number such that $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$ then a,b,c are the direction ratio of the line

Exceptionalcases:

1. If one of the direction cosine of a line L , say l = 0 and m $\neq$ 0, n $\neq$ 0 then direction ration of L are given by (0,b,c) where $\frac{b}{m}=\frac{c}{n}$ abd b and c are nonzero real number.
2. If two direction cosine are zero l = m = 0 and n $\neq$ 0 then obviously n = $\pm$1 and the direction ratio are (0,0,c) , c $\in$ R, c $\neq$ 0.

Finding Direction Cosine from Direction Ratio

If a,b,c are direction ration of a line then its direction cosine are given by

$l=\frac{a}{\pm\sqrt{a^2+b^2+c^2}}$, $m=\frac{b}{\pm\sqrt{a^2+b^2+c^2}}$, $n=\frac{c}{\pm\sqrt{a^2+b^2+c^2}}$

Direction Ratio of the line segment joining two points : $\frac{(x_2-x_1)}{\cos\alpha}=\frac{(y_2-y_1)}{\cos\beta}=\frac{(z_2-z_1)}{\cos\gamma}$

Angle between two lines with given Direction ratio

If L_1 *and* L_2 are not parallel lines having direction cosine $<l_1,m_1,n_1>$ and $<l_2,m_2,n_2>$ and θ is the measure of angle between them then $\cos\theta=l_1l_2+m_1m_2+n_1n_2$

Proof :

Consider the ray $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ such that $\underset{OP}{\rightarrow} \parallel L_1$ and $\underset{OQ}{\rightarrow} \parallel L_2$. $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ are taken in such a way that m<POQ = θ and direction cosine of $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ are respectively $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$. Let P andQ have a co-ordinate respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

In ΔOPQ $\cos\theta = \frac{OP^2+OQ^2-PQ^2}{2\,OP\,.\,OQ}$

$= \frac{(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)-\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{2\,OP\,.\,OQ}$

$= \frac{x_1x_2+y_1y_2+z_1z_2}{OP\,.\,OQ} = \frac{x_1}{OP}\frac{x_2}{OQ} + \frac{y_1}{OP}\frac{y_2}{OQ} + \frac{z_1}{OP}\frac{z_2}{OQ} = l_1l_2 + m_1m_2 + n_1n_2$

Note that L_1 *and* L_2 is perpendicular then $\cos\theta = 0$.

1. Thus the line with direction cosine $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$ are perpendicular only if $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
2. If $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are direction ratio of L_1 *and* L_2 and θ measures the angle between them then
   $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Lines with direction ratio $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are perpendicular if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

3. Since parallel lines have same direction cosine it follows from the definition of direction ratio that lines with direction ratio $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are parallel if and only if
   $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Example :

Find the direction cosine of the line which is perpendicular to the lines whose direction ratios are <1,-2,3> and <2,2,1>

Solution :

Let l, m, n be the direction cosine of the line which is perpendicular to the given lines. Then we have

l.1+m.(-2)+n.3 = 0 and l.2+m.2+n.1 =0

By cross multiplication we have

$$\frac{l}{-2-6} = \frac{m}{6-1} = \frac{n}{2+4}$$

Or, $\frac{l}{-8} = \frac{m}{5} = \frac{n}{6} = k \quad then\, l = -8k;\ m = 5k; n = 6k$

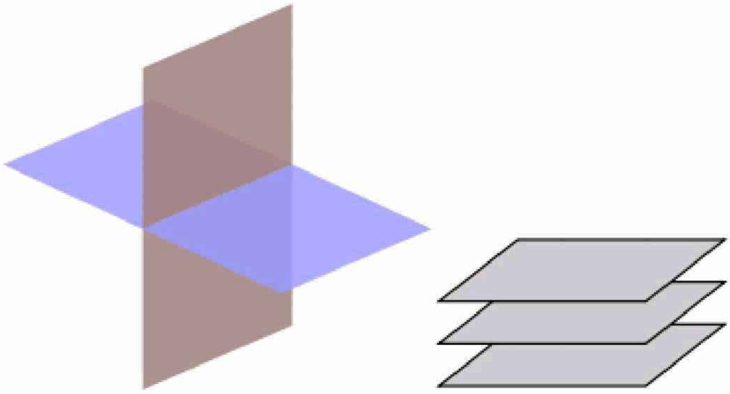
$l^2 + m^2 + n^2 = 1 \quad => (64 + 25 + 36)k^2 = 1$

Or $k^2 = \frac{1}{125} \quad => k = \frac{1}{5\sqrt{5}}$

$\therefore l = \frac{8}{5\sqrt{5}}; m = \frac{1}{\sqrt{5}}\ ; n = \frac{6}{5\sqrt{5}}$

Plane

In mathematics, a **plane** is a flat, two-dimensional surface. A plane is the two-dimensional analogue of a point (zero-dimensions), a line (one-dimension) and a solid (three-dimensions). Planes can arise as subspaces of some higher-dimensional space, as with the walls of a room, or they may enjoy an independent existence in their own right,



Properties

The following statements hold in three-dimensional Euclidean space but not in higher dimensions, though they have higher-dimensional analogues:

- Two planes are either parallel or they intersect in a line.
- A line is either parallel to a plane, intersects it at a single point, or is contained in the plane.
- Two lines perpendicular to the same plane must be parallel to each other.
- Two planes perpendicular to the same line must be parallel to each other.

Point-normal form and general form of the equation of a plane

In a manner analogous to the way lines in a two-dimensional space are described using a point-slope form for their equations, planes in a three dimensional space have a natural description using a point in the plane and a vector (the normal vector) to indicate its "inclination".

Specifically, let $\mathbf{r}_0$ be the position vector of some point $P_0 = (x_0, y_0, z_0)$, and let $\mathbf{n} = (a, b, c)$ be a nonzero vector. The plane determined by this point and vector consists of those points P, with position vector $\mathbf{r}$, such that the vector drawn from P_0 to P is perpendicular to $\mathbf{n}$. Recalling that two vectors are perpendicular if and only if their dot product is zero, it follows that the desired plane can be described as the set of all points $\mathbf{r}$ such that

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

(The dot here means a dot product, not scalar multiplication.) Expanded this becomes

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0,$$

which is the *point-normal* form of the equation of a plane.[3] This is just a linear equation:

$$ax+by+cz+d=0,\ \text{where}\ d=-(ax_0+by_0+cz_0).$$

Conversely, it is easily shown that if *a*, *b*, *c* and *d* are constants and *a*, *b*, and *c* are not all zero, then the graph of the equation

$$ax+by+cz+d=0,$$

is a plane having the vector $\mathbf{n}=(a,b,c)$ as a normal.[4] This familiar equation for a plane is called the *general form* of the equation of the plane.[5]

Example :

Find the equation of the plane through the point (1,3,4), (2,1,-1) and (1,-4,3).

Ans :

Any plane passing through (1,3,4) is given by

A(x-1) + B(y-3) + C(z-4) = 0 ….(1)

Where A,B,C are direction ratio of the normal to the plane.

Since the passes through(2,1,-1) and (1,-4,3) we have

A(2-1) +B(1-3)+C(-1-4) = 0

Or A -2B-5C = 0 …..(1)

A(1-1)+B(-4-3)+C(3-4) = 0

Or A.0 + B(-7)+C(-1) = 0

Or -7B-C = 0 …..(2)

By Type equation here.cross multiplication we get

$$\frac{A}{(-2)(-1)-(-5)(-7)}=\frac{B}{(-5)0-(1)(-1)}=\frac{C}{1(-7)-0.(-2)}$$

Or, $\frac{A}{-33}=\frac{B}{1}=\frac{C}{-7}$

Hence the direction ratio of the normal to the plane are 33,-1,7 and putting these values in (1), the equation of the required plane is

33(x-1)-1(y-3)+7(z-4) = 0

Or 33x – y +7z-58 = 0

Equation Of plane in normal form

Fig

Let p be the length of the perpendicular $\overline{ON}$ from the origin on the plane and let <l,m,n> be its direction cosines. Then the co-ordinate of the foot of the perpendicular N are (lp,mp,np).

If P(x,y,z) be any point on the plane then the direction ratio of $\overline{NP}$ are (x-lp,y-mp,z-np). Since $\overline{ON}$ is perpendicular to the plane it is also perpendicular to $\bar{N}P$

Hence

L(x – lp) + m(y – mp)+n(z –np) = 0

Or, lx +my+nz = $(l^2 + m^2 + n^2)p$

Or lx+my+nz =p

Example :

Obtain the normal form of equation of the plane 3x+2y+6z+1 = 0 and find the direction cosine and length of the perpendicular from the origin to this plane.

Solution :

The direction ratios of the normal to the plane are <3,2,6> and hence the direction cosines are

$< \frac{3}{\pm\sqrt{9+4+36}}, \frac{2}{\pm\sqrt{9+4+36}}, \frac{6}{\pm\sqrt{9+4+36}} >$

Length of the perpendicular from origin is

P = $\frac{-D}{\pm\sqrt{A^2+B^2+C^2}} = \frac{-1}{\pm\sqrt{9+4+36}} = \frac{1}{7}$

$(\because D \text{ is positive we choose negative before the radical sign to make } p > 0$

The equation of plane in normal form is

$$\frac{A}{-\sqrt{A^2+B^2+C^2}}x + \frac{B}{-\sqrt{A^2+B^2+C^2}}y + \frac{C}{-\sqrt{A^2+B^2+C^2}}z + \frac{D}{-\sqrt{A^2+B^2+C^2}} = 0$$

Or $\frac{3}{-7}\,x + \frac{2}{-7}\,y + \frac{6}{-7}\,z + \frac{1}{-7} = 0$

Distance Of a point from a plane

Fig

Let P(x_1, y_1, z_1) be a given point and Ax+By+Cz+D = 0 be the equation of a given plane. Draw $\overline{QN}$ normal to the plane at Q and $\overline{PM}$ perpenticular to $\overline{QN}$. Join $\overline{PQ}$ .If R be the foot of the perpentcular drawn from the point Pto the given plane, then

D= PR=QM= projection of $\overline{PQ}$ on $\overrightarrow{QN}$. $\overrightarrow{QN}$ being normal to the given plane Ax+By+Cz+D = 0 the direction ratio of $\overrightarrow{QN}$ are <A,B,C> and the direction cosines are

$$\langle \frac{A}{\pm\sqrt{A^2+B^2+C^2}}, \frac{B}{\pm\sqrt{A^2+B^2+C^2}}, \frac{C}{\pm\sqrt{A^2+B^2+C^2}} \rangle$$

$\therefore d = projection\ of\ line\ segment\ \overline{PQ}\ on\ \overrightarrow{QN}\ .\ \overrightarrow{QN}$

$$= \frac{A}{\pm\sqrt{A^2+B^2+C^2}}\,(x_0 - \alpha) + \frac{B}{\pm\sqrt{A^2+B^2+C^2}}\,(y_0 - \beta) + \frac{C}{\pm\sqrt{A^2+B^2+C^2}}\,(z_0 - \gamma)$$

$$= \frac{A(x_0 - \alpha) + B(y_0 - \beta) + C(z_0 - \gamma)}{\pm\sqrt{A^2+B^2+C^2}}$$

$$= \frac{Ax_0 + By_0 + Cz_0 - (A\alpha + B\beta + C\gamma)}{\pm\sqrt{A^2+B^2+C^2}}$$

Now (α, β, γ) $lies\ on\ the\ given\ plane$ $(A\alpha + B\beta + C\gamma + D = 0$ hence $(A\alpha + B\beta + C\gamma = -D$

Thus

$$d = \frac{Ax_0 + By_0 + Cz_0 + D)}{\pm\sqrt{A^2+B^2+C^2}}$$

The sign of the denominator chosen accordingly so as to make the whole quantity positive. In particular the distance of the plane from the origin is given by

$$\frac{D}{\pm\sqrt{A^2+B^2+C^2}}$$

Example

Find the distance d from the point P(7,5,1) to the plane 9x+3y-6z-2 = 0

Solution :

Let R be any point of the plane. The scalar projection of vector $\overline{RP}$ on a vector perpendicular to the plane gives the required distance. The scalar projection is obtained by taking the dot project of $\overline{RP}$ and a unit vector normal to the plane. The point (1,0,0) is in the plane and using this point for R, we have $\overline{RP}$ = 6i+5j+k

N = $\pm\frac{2i+3j-6k}{7}$ is a unit vector normal to the plane. Hence

N . $\overline{RP}$ $= \pm\frac{12+15-6}{7} = \pm\frac{21}{7}$ we choose the ambiguous sign + in order to have a positive result. Thus we get d = 3.

Dihedral angle(Angle Between two planes)

Given two intersecting planes described by

$$\Pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and}$$

$$\Pi_2 : a_2x + b_2y + c_2z + d_2 = 0,$$

the dihedral angle between them is defined to be the angle α between their normal directions:

$$\cos\alpha = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Example : Find the angle θ between the plane 4x-y+8z+7 =0 and x + 2y-2z+5 = 0

Solution :

The angle between two plane is equal to the angle between their normals. The vectors

$N_1 = \frac{4i-j+8k}{9} \quad N_2 = \frac{i+2j-2k}{3}$

Are unit vectors normal to the given planes. The dot product yield

$\cos\theta = N_1 . N_2 = -\frac{14}{27}$ or $\theta = 121^o$

Equation Of plane passing through three given point

Let (x_1, y_1, z_1) , (x_2, y_2, z_3) and (x_3, y_3, z_3) be three given points and the required plane be Ax+By+Cz+D =0……………….. (1)

Since it passes through (x_1, y_1, z_1) we have

Ax_1 + By_1+ Cz_1 = 0 ………(2)

Subtracting eq (2) from (1) we have

A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 …………(3)

Since this plane also passes through (x_2, y_2, z_3) and (x_3, y_3, z_3) we have

$A(x_2 - x_1) + B\,(y_2 - y_1) + \;C(z_2 - \;z_1) = 0$

And

$A(x_3 - x_1) + B\,(y_3 - y_1) + \;C(z_3 - \;z_1) = 0$ ………… (5)

Eliminating A,B, C from eq (3),(4) and (5) we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Which is the equation of the plane

Corollary 1:

If the plane makes the intercepts a,b,c on the co-ordinate axes $\overleftrightarrow{OX}, \overleftrightarrow{OY}, \overleftrightarrow{OZ}$ respectively then the plane passes through the point (a,0,0) , (0.b,0) and (0,0,c). Hence the equation (6) gives

$$\begin{vmatrix} x - a & y - 0 & z - 0 \\ 0 - a & b - 0 & 0 - 0 \\ 0 - a & 0 - 0 & c - 0 \end{vmatrix} = \begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \text{bc(x-a) +yac+zab = o}$$

Dividing both side by abc we get $\frac{(x-a)}{a} + \frac{y}{b} + \frac{z}{c} = 0$

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Example :

Find the equation of the plane determined by the point $P_1(2,3,7), P_2(2,3,7), P_3(2,3,7)$

Solution :

A vector which is perpendicular to two sides of tringle $P_1P_2P_3$ is normal to the plane of the tringle. To find the vector we write

$\overline{P_1P_2} = 3i + 2j - 5k$ $\quad P_1P_3 = i + j - k$ $\quad$ N = Ai+Bj+Ck

The coeff A,B,C are to be found so that N is perpendicular to each of the vector Thus

N. P_1P_2 = 3A+2B-5C = 0

N. P_1P_3 = A+B-C = 0

These equation gives A = 3C and B = -2C. Choosing C =1 we have N = 3i-2j+k. Hence tha plane 3x-2y+z+D = 0 is normal to N and passing through the points if D =-7

Hence the equation is 3x-2y+z-7 = 0.

Alternate :

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = \begin{vmatrix} x-2 & y-3 & z-7 \\ 5-2 & 5-3 & 2-7 \\ 3-2 & 4-3 & 6-7 \end{vmatrix} == \begin{vmatrix} x-2 & y-3 & z-7 \\ 3 & 2 & -5 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

=(x-2)(-2+5)-(y-3)(-3+5)+(z-7)(3-2) =0

= 3x-6-2y+6+z-7= 0

Or 3x -2y+z -7 = 0

Exercise

1. Write the equation of the plane perpendicular to N = 2i-3j+5k and passing through the point (2,1,3) Ans 2x-3y+5z-6 = 0
2. Parallel to the plane 3x-2y-4z = 5 and passing through (2,1,-3) Ans 3x-2y-4z-16 = 0
3. Passing through the (3,-2,-1)(-2,4,1)(5,2,3) Ans 2x+3y-4z-4 = 0
4. Find the perpendicular distance from 2x-y+2z+3 = 0 (1,0,3) Ans :$\frac{11}{3}$

5. Find the perpendicular distance from 4x-2y+z-2 =0 (-1,2,1) Ans : $\frac{9}{\sqrt{21}}$

6. Find the cosine of the acute angle between each pair of plane 2x+2y+z-5 = 0 , 3x-2y+6z+5 = 0

Ans : $\frac{8}{21}$

7. Find the cosine of the acute angle between each pair of plane 4x-8y+z-3 = 0, 2x+4y-4z+3 = 0

Ans : $\frac{14}{27}$